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Nonlinear vibration and dynamic stability analysis of composite plates

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ABSTRACT

Large amplitude flexural vibration characteristics of composite plates under transverse harmonic pressure or periodic in-plane load are investigated here using the shear deformable finite element method. The nonlinear stiffness matrix is formulated based on von Kármán's assumptions to obtain the stiffness interaction between the in-plane and bending degrees of freedom. Further, the flexural motion of the plate is assumed to be harmonic and the in-plane movement is assumed to be periodic. The nonlinear matrix amplitude equation is obtained by employing Galerkin's method. The coupled nonlinear matrix amplitude equation (in-plane motion is coupled with flexural motion) is solved to obtain (1) nonlinear free flexural vibration frequencies of isotropic and composite plates with different in-plane boundary conditions, (2) flexural vibration amplitudes of such plates under transverse harmonic pressure or periodic in-plane load. Finally, the time history analysis is carried out to understand the steady-state or unsteady nature of the flexural vibration under different loading and boundary condition.

1. Introduction

Nonlinear dynamics of plate like structures is a complex phenomenon involving both transverse displacement and in-plane strain. The stiffness interaction (coupling) between the in-plane and bending degrees of freedoms are generally expressed through von Kármán's strain-displacement assumptions. Several analytical and numerical studies on the large amplitude free flexural vibration behavior of isotropic or composite plates are reported in the literature [1,2]. In the analytical investigations, the equation of motion for the nonlinear vibration of plates was reduced to a single nonlinear differential equation in time by assuming single space-mode, which was solved by classical elliptic function approach [3,4], perturbation method [5] and Runge–Kutta integration [6]. Numerical techniques, such as finite element method, overcome the limitations of assumed space-mode. However, the selection of appropriate time function or determination of a steady-state periodic solution of the differential equations with quadratic and cubic nonlinearity is a challenge to the researchers working in the area of nonlinear dynamics of composite plates. The notable numerical contributions on this subject are the use of harmonic balance method [7,8], multimode approach [9] and incremental Hamilton's principle [10]. A thorough review of the existing literature [11,12] reveals that the relationship between the nonlinear flexural vibration frequency (ω_{NL}) and maximum vibration amplitude ($w_{\max}$) for the free vibration of rectangular plates with immovable in-plane

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boundary condition are extensively investigated. However, studies on the influences of in-plane movements on the free and forced vibration characteristics of plates with movable in-plane boundary conditions are limited in the literature.

Large amplitude flexural vibration may occur for such composite plates not only under transverse load, but also under periodic in-plane load for certain combinations of load amplitude and disturbing frequency. This loss of stability (i.e., the transverse vibration) of plates under dynamic in-plane load is known as dynamic instability [13,14]. The primary dynamic stability region of such plates which occur for the excitation frequency near $2\omega_L$ (ω_L is the lowest flexural vibration frequency of the plate) has long been studied using linear structural theory [15–18]. However, understanding of the nonlinear dynamic instability characteristics of such plates requires an in-depth knowledge of the nonlinear flexural vibration frequencies and their interaction with in-plane forcing frequency.

Few interesting studies on the effect of vibration amplitude on nonlinear dynamic stability regions of composite plates are available in the literature [19–24]. Librescu and Thangjitham [19] examined the nonlinear Mathiew–Hill equation obtained from the assumed space-mode analytical solution of the plate and studied the effect of movable boundaries on the dynamic instability region and frequency-amplitude response curves of shear deformable composite plates. Bhimaraddi [20] employed the method of multiple-scales, Marin et al. [21] applied harmonic balance method, while Wu and Shih [22] used incremental harmonic balance method to obtain an approximate periodic solution for nonlinear flexural vibration amplitudes of laminated composite plates subjected to periodic in-plane loads. However, the steady-state or transient nature of the flexural vibration and the effect of initial condition were not brought out in the above works due to the limitations of their solution methodology. Analytical and experimental studies [13] indicate the existence of beats or unsteady nature of the flexural vibration of plates under periodic in-plane load. Most recently, Ganapathi et al. [23,24] carried out nonlinear dynamic response analysis to understand the parametric resonance characteristics of movable composite plates subjected to periodic in-plane load and reported the existence of beats, i.e., the unsteady nature of vibration. However, further studies are required to understand the effect of initial condition on the steady-state or unsteady nature of the flexural vibration of composite plates under periodic in-plane load.

In the present paper, a four-noded shear flexible quadrilateral high precision plate bending element [25] is employed to analyze the large amplitude flexural vibration characteristics of thin isotropic and composite plates under transverse harmonic pressure or periodic in-plane load. The nonlinear stiffness matrix is formulated based on von Kármán's assumptions to obtain the stiffness interaction between the in-plane and bending degrees of freedom. Suitable time functions for the in-plane and bending degrees of freedom are assumed and a nonlinear matrix amplitude equation is formulated using Galerkin's method [2]. The coupled nonlinear matrix amplitude equation is solved to obtain the nonlinear free flexural vibration frequencies of isotropic and composite plates with different boundary conditions. Thereafter, the frequency-amplitude relationships for the flexural vibration of such plates under transverse harmonic pressure or periodic in-plane load are evaluated using the same matrix-amplitude equation. Finally time history analysis of such plates is carried out using Newmark's time integration technique to investigate the steady-state or transient nature of the flexural vibration under periodic transverse or in-plane load.

2. Finite element formulations

The displacement components at a generic point (x, y, z) of a shear deformable rectangular plate can be expressed as [25]

$$u(x,y,z) = u_0(x,y) + z\{-w_{,x} + \gamma_{xz}(x,y)\}$$

$$v(x,y,z) = v_0(x,y) + z\{-w_{,y} + \gamma_{yz}(x,y)\}$$

$$w(x,y,z) = w_0(x,y) \tag{1}$$

Here, u_0, v_0, w are the mid-surface displacements; γ_{xz} and γ_{yz} are the rotations due to shear; $(\ )_{,x}$ and $(\ )_{,y}$ represent the partial differentiation with respect to x and y; $\phi_x = -w_{,x} + \gamma_{xz}(x, y)$ and $\phi_y = -w_{,y} + \gamma_{yz}(x, y)$ are the nodal rotations.

A four-noded rectangular high precision plate bending element [25] with the following complete cubic polynomial shape functions for the in-plane and lateral displacements (u_0, v_0, w_0) and linear polynomial shape functions for the shear strains (γ_{xz}, γ_{yz}) is employed here:

$$u_0 = [1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, x^3y^2, x^2y^3, x^3y^3]\{c_i\}, i = 1, 16$$

$$v_0 = [1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, x^3y^2, x^2y^3, x^3y^3]\{c_i\}, i = 17, 32$$

$$w_0 = [1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, x^3y^2, x^2y^3, x^3y^3]\{c_i\}, i = 33, 48$$

$$\gamma_x = [1, x, y, xy]\{c_i\}, i = 49, 52$$

$$\gamma_y = [1, x, y, xy]\{c_i\}, i = 53, 56 \tag{2}$$

where c_i are constants and are expressed in terms of nodal displacements (u_0, $u_{0,x}$, $u_{0,y}$, $u_{0,xy}$, v_0, $v_{0,x}$, $v_{0,y}$, $v_{0,xy}$, w, $w_{,x}$, $w_{,y}$, $w_{,xy}$, γ_{xz} and γ_{yz}) in the finite element discretization.

Following standard procedure (minimization of potential energy), the equation of equilibrium of the plate under lateral pressure and in-plane load may be written as

$$\begin{bmatrix} M_{mm} & 0 \\ 0 & M_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_m \\ \ddot{\delta}_b \end{Bmatrix} + \begin{bmatrix} K_{mm} & K_{mb} \\ K_{bm} & K_{bb} \end{bmatrix} \begin{Bmatrix} \delta_m \\ \delta_b \end{Bmatrix} + \begin{bmatrix} 0 & N_1(w) \\ N_2(w) & N_3(w,w) \end{bmatrix} \begin{Bmatrix} \delta_m \\ \delta_b \end{Bmatrix} = \begin{Bmatrix} N(t) \\ q(t) \end{Bmatrix} \tag{3}$$

Here, ***M*** and ***K***, are the mass and linear stiffness matrix respectively, $\boldsymbol{N_1}$, $\boldsymbol{N_2}$ and $\boldsymbol{N_3}$ are nonlinear stiffness matrices (depends on transverse displacement w); Subscripts 'm' and 'b' correspond to membrane (u_0, v_0) and bending (w, γ_{xz}, γ_{yz}) components of the degrees of freedom and the corresponding mass and stiffness matrices respectively. $\boldsymbol{q}(\boldsymbol{t})$ and $\boldsymbol{N}(\boldsymbol{t})$ are the load vectors corresponding to uniformly distributed transverse load and in-plane edge load, respectively.

3. Solution procedure

The displacement components for the nonlinear vibration of symmetrically laminated composite plates under transverse harmonic pressure $q_0 \sin\theta t$ or periodic in-plane load $N_d \sin^2\theta t$ are assumed to be of the form

$$\delta(t) = \{u_0 \sin^2\theta t,\ \ v_0 \sin^2\theta t,\ \ w\sin\theta t,\ \ \gamma_{xz}\sin\theta t,\ \ \gamma_{yz}\sin\theta t\}^{\mathrm{T}} \tag{4}$$

substituting the assumed solution (4) in the governing Eq. (3) one gets

$$\begin{bmatrix} K_{mm} & K_{mb}+N_1\sin\theta t \\ K_{bm}+N_2\sin\theta t & K_{bb}+N_3\sin^2\theta t \end{bmatrix} \begin{Bmatrix} \delta_m\sin^2\theta t \\ \delta_b\sin\theta t \end{Bmatrix} - \theta^2 \begin{bmatrix} -2M_{mm} & 0 \\ 0 & M_{bb} \end{bmatrix} \begin{Bmatrix} \delta_m\cos 2\theta t \\ \delta_b\sin\theta t \end{Bmatrix} - \begin{Bmatrix} N_d\sin^2\theta t \\ q_0\sin\theta t \end{Bmatrix} = \begin{Bmatrix} R_u \\ R_w \end{Bmatrix} \tag{5}$$

Since Eq. (5) does not satisfy the equilibrium equation at all the points, taking the weighted residual along the path $\int_0^{T/4}\{R_u\}\sin^2\theta t = \{0\}$ and $\int_0^{T/4}\{R_w\}\sin\theta t = \{0\}$ the following matrix-amplitude equation is obtained

$$\begin{bmatrix} \frac{3}{4}K_{mm} & \frac{8}{3\pi}K_{mb}+\frac{3}{4}N_1 \\ \frac{8}{3\pi}K_{bm}+\frac{3}{4}N_2 & K_{bb}+\frac{3}{4}N_3 \end{bmatrix} \begin{Bmatrix} \delta_m \\ \delta_b \end{Bmatrix} - \theta^2 \begin{bmatrix} -M_{mm} & 0 \\ 0 & M_{bb} \end{bmatrix} \begin{Bmatrix} \delta_m \\ \delta_b \end{Bmatrix} = \begin{Bmatrix} \frac{3}{4}N_d \\ q_0 \end{Bmatrix} \tag{6}$$

This matrix-amplitude Eq. (6) is employed to study the free and forced vibration characteristics of isotropic and composite plates as follows:

3.1. Nonlinear free flexural vibration

In the case of free flexural vibration (q_0=N_d=0), the matrix amplitude Eq. (6) is solved iteratively as explained in Ref. [2] to obtain the backbone curves i.e., the frequency-amplitude relationships of isotropic and composite plates.

3.2. Nonlinear forced vibration under transverse harmonic pressure $q_0 \sin\theta t$

For the case of forced vibration under transverse harmonic load $q_0\sin\theta t$, the above matrix amplitude equation (N_d=0) is solved by Newton–Raphson technique to obtain the steady-state flexural vibration amplitude $\{\delta_m, \delta_b\}^{\mathrm{T}}$ (the maximum nodal displacements) of isotropic and composite plates corresponding to non-dimensional excitation frequency θ/ω_L and load parameter q_0.

3.3. Primary dynamic stability under in-plane load $N_d\sin^2\theta t$

Similarly, for the case of periodic in-plane load of the form $N_d\sin^2\theta t$, Eq. (6) is solved by Newton–Raphson technique to obtain the steady-state flexural vibration amplitude of the plate $\{\delta_m, \delta_b\}^{\mathrm{T}}$ corresponding to non-dimensional excitation frequency θ/ω_L and load parameter N_d/N_{cr}. Here, N_{cr} is the static buckling load.

3.4. Primary dynamic stability under in-plane load $N_0 \pm N_1\cos 2\theta t$

In general, the periodic in-plane load and the displacement components are expressed as

$$N(t) = N_0 \pm N_1\cos 2\theta t \quad \text{or} \quad N(t) = N_s \pm N_d\sin^2\theta t$$

$$\delta(t) = \{u_s + u_d\sin^2\theta t,\ \ v_s + v_d\sin^2\theta t,\ \ w\sin\theta t,\ \ \gamma_{xz}\sin\theta t,\ \ \gamma_{yz}\sin\theta t\}^{\mathrm{T}}$$

$$\{\delta_m(t)\} = \{\delta_s\} + \{\delta_d\}\sin^2\theta t \tag{7}$$

Here, "*s*" and "*d*" represents the static and dynamic components of the in-plane displacements (u_0, v_0). Substituting the above displacement components into the equilibrium Eq. (3) and taking the weighted residual as before, the matrix-amplitude equation is modified as

$$[K_{mm}]\{\delta_s\} = \{N_s\} \tag{8a}$$

$$\begin{bmatrix} \frac{3}{4}K_{mm} & \frac{8}{3\pi}K_{mb} + \frac{3}{4}N_1 \\ \frac{8}{3\pi}K_{bm} + \frac{3}{4}N_2 & K_{bb} + \frac{3}{4}N_3 \end{bmatrix} \begin{Bmatrix} \delta_d \\ \delta_b \end{Bmatrix} - \theta^2 \begin{bmatrix} -M_{mm} & 0 \\ 0 & M_{bb} \end{bmatrix} \begin{Bmatrix} \delta_d \\ \delta_b \end{Bmatrix} = \begin{Bmatrix} \pm\frac{3}{4}N_d \\ 0 \end{Bmatrix} - [N_2]\{\delta_s\} \tag{8b}$$

Eq. (8) is solved for $\{\delta_s\}$, $\{\delta_d\}$ and $\{\delta_b\}$ to obtain the steady-state flexural vibration amplitude $\{\delta_m, \delta_b\}^{\mathrm{T}}$ of the composite plates corresponding to non-dimensional excitation frequency θ/ω_L and load parameters N_s/N_{cr} and N_d/N_{cr}.

3.5. Time history analysis under periodic transverse or in-plane load

Further, the governing Eq. (3) is solved with Newmark's time integration technique starting from different initial conditions (in particular $\{\delta_m, \delta_b\}^{\mathrm{T}}$ obtained from the corresponding matrix-amplitude equation) to investigate the steady-state or unsteady-state nature of flexural vibration under transverse harmonic pressure or periodic in-plane load. This is an implicit time integration method, where the nonlinear stiffness matrix is continuously updated until the following equation [27] is satisfied up to desired accuracy

$$[K_L + K_{NL} + a_0 M]\{\delta_{t_i+\Delta t}\} = \{F_{t_i+\Delta t}\} + M(a_0\delta_{t_i} + a_2\dot{\delta}_{t_i} + a_3\ddot{\delta}_{t_i}) \tag{9}$$

here $\boldsymbol{K}_{NL}$ is the nonlinear stiffness matrix. The constants are defined as $a_0 = 1/\gamma\Delta t^2$, $a_2 = 1/\gamma\Delta t$ and $a_3 = (1/2\gamma) - 1$, where $\gamma = 0.25\Delta t$ is the time step.

4. Results and discussions

Nonlinear vibration and dynamic stability characteristics of thin laminated composite plates are studied here. The material properties, used in the present analysis are

$$E_L/E_T = 40.0,\ G_{LT}/E_T = 0.6, G_{TT}/E_T = 0.5,\ v_{LT} = 0.25; E_T = 100000.0 \text{ and } \rho = 1.0$$

where E, G, v and ρ are Young's modulus, shear modulus, Poisson's ratio and density. Subscripts L and T represent the longitudinal and transverse directions respectively with respect to the fibers. All the layers are of equal thickness. For isotropic plate, Poisson's ratio (v) is taken as 0.3. Fiber orientation is measured from X-axis. The different boundary conditions considered in the present analysis are

Simply supported cases: w=0 at x=0, a and y=0, b

In addition to the above constraints on transverse displacement (w), the following four different in-plane boundary conditions are considered here:
Immovable (SS1) u_0=v_0=0 at x =0, a and y=0, b
Movable (SS2) u_0=0 at x=a/2; v_0=0 at y=b/2

Table 1
Comparison of nonlinear frequency ratio (ω_{NL}/ω_L) of immovable isotropic square plates (a=b; a/h=1000).

w/h	0.2	0.4	0.6	0.8	1
Immovable simply supported (*SS*1)					
Present study					
5 × 5 mesh	1.01962	1.07649	1.16557	1.28067	1.41597
7 × 7 mesh	1.01966	1.07662	1.16584	1.28110	1.41655
9 × 9 mesh	1.01966	1.07665	1.16589	1.28003	1.41666
Analytical integration [3]	1.0195	1.0757	1.1625	1.2734	1.4024
Spline finite strip method [26]	1.0197	1.0768	1.1662	1.2813	1.41729
Multi-mode approach [9]	1.0195	1.0765	1.1658	1.2796	1.4163
Immovable clamped (*CC*2)					
Present study					
5 × 5 mesh	1.00735	1.02909	1.06439	1.11201	1.17054
7 × 7 mesh	1.00725	1.02869	1.06344	1.11021	1.16759
9 × 9 mesh	1.00723	1.02860	1.06322	1.10981	1.16690
Elliptic integral [4]	1.0078	1.0326	1.069	1.1173	1.1757
Spline finite strip method [26]	1.0073	1.0287	1.0633	1.1101	1.1671
FE contd. method [10]	1.0073	1.0291	1.0648	1.1138	1.1762

Partially movable (SS3) v_0=0, at x=0, a; u_0=0, at y=0, b
Partially movable (SS4) u_0=v_0=0 at x=a and y=b;
u_0=constant at x=0, v_0=constant at y=0

Clamped cases: w=0, $w_{,x}$=0 at x=0, a; w=0, $w_{,y}$=0 at y=0, b

In addition to the above constraints on transverse displacement (w), the following two different in-plane boundary conditions are considered here

Immovable (CC1) u_0=v_0=0 at x =0, a and y=0, b
Movable (CC2) u_0=0 at x=$a/2$; v_0=0 at y=$b/2$

Before proceeding for the detailed parametric study, the efficacy of the present formulation is tested herein by studying the nonlinear free flexural vibration frequencies of immovable simply supported (SS1) and clamped (CC1) isotropic plates for which several numerical results are available in the literature. The matrix amplitude Eq. (6) with q_0=N_0=0 is solved iteratively to obtain the nonlinear frequency (ω_{NL}) corresponding to maximum amplitude ($w_{\max}$) of the plate. The variation of nonlinear frequency ratio (ω_{NL}/ω_L; ω_L is the linear frequency) with non-dimensional maximum amplitude ($w_{\max}/h$) is evaluated for simply supported and clamped thin (a/h=1000) square plates and are shown in Table 1 along with the published results. It is observed that the present results for the nonlinear frequency of immovable isotropic plates are in close agreement with the available solutions. Further, a 9×9 mesh is found to be sufficient to model the full plate.

Table 2
The nonlinear frequency ratios (ω_{NL}/ω_L) of thin square plates (a/b=1; a/h=100) with different boundary conditions.

w/h	Boundary condition					
	SS1	SS2	SS3	SS4	CC1	CC2
Isotropic plate						
0.2	1.01967	1.00265	1.00637	1.00516	1.00728	1.00234
0.4	1.07669	1.01052	1.02518	1.02049	1.02878	1.00928
0.6	1.16597	1.02336	1.05559	1.04559	1.06363	1.02060
0.8	1.28131	1.0408	1.09640	1.07959	1.11055	1.03594
1.0	1.41684	1.06237	1.14618	1.12239	1.16808	1.05487
1.2	1.56783	1.08756	1.20331	1.17249	1.23474	1.07691
Cross-ply [0°/90°/0°/90°/0°] *plate*						
0.2	1.03147	1.00121	1.00165	1.01057	1.00847	1.00076
0.4	1.12099	1.00482	1.00658	1.04169	1.03347	1.00303
0.6	1.25723	1.01079	1.01473	1.09167	1.07390	1.00678
0.8	1.42805	1.01906	1.02599	1.15823	1.12820	1.01196
1.0	1.62368	1.02952	1.04023	1.23885	1.19461	1.01850
1.2	1.83697	1.04208	1.05729	1.33111	1.27141	1.01850
Angle-ply [45°/−45°/45°/−45°/45°] *plate*						
0.2	1.01473	1.00085	1.00796	1.00082	1.00717	1.00090
0.4	1.05776	1.00339	1.03135	1.00327	1.02834	1.00358
0.6	1.12601	1.00755	1.06885	1.00734	1.06258	1.00796
0.8	1.21542	1.01324	1.11848	1.01301	1.10855	1.01393
1.0	1.32185	1.02033	1.17837	1.02024	1.16474	1.02133
1.2	1.44162	1.02871	1.24629	1.02900	1.22969	1.03000

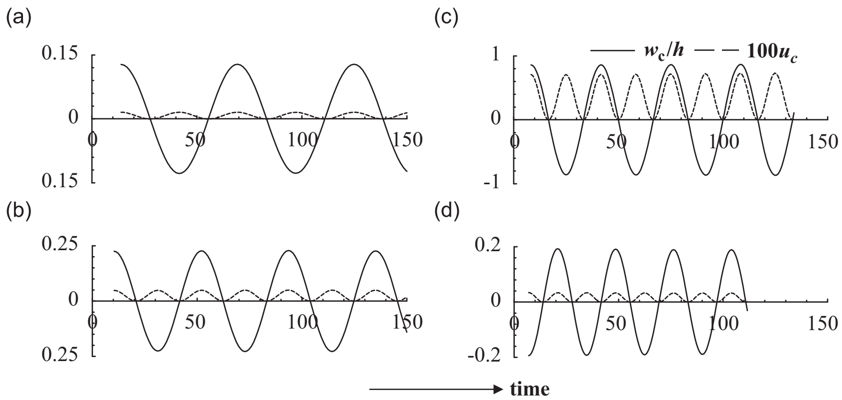

Fig. 1. Steady-state flexural vibration of an isotropic square plate with movable simply supported boundary condition (SS2) under uniformly distributed transverse harmonic pressure $q_0 \sin\theta t$ (a/h=100; q_0=0.20D/a^3). (a) θ=0.6ω_L, (b) θ=0.8ω_L, (c) θ=1.0ω_L, (d) θ=1.2ω_L.

4.1. Nonlinear free flexural vibration

Next, the backbone curves, i.e., the relationship between the nonlinear frequency ratio (ω_{NL}/ω_L) and non-dimensional maximum amplitude ($w_{\max}/h$) of isotropic, cross-ply [0°/90°/0°/90°/0°] and angle-ply [45°/−45°/45°/−45°/45°] thin square plates (a/b=1; a/h=100) are studied for all the six boundary conditions (SS1, SS2, SS3, SS4, CC1 and CC2) in Table 2. It is observed from Table 2 that the nonlinear flexural vibration frequencies (ω_{NL}) of isotropic plates increase with the increase of vibration amplitude (w/h) for all the six boundary conditions considered in the present investigation. The degree of hardening nonlinearity (i.e., increase of frequency with amplitude) is more for simply supported plates compared to those of clamped plates and further depends on the in-plane boundary conditions. Immovable simply supported (SS1) or clamped (CC1) plates have highest nonlinear flexural vibration frequencies, whereas, corresponding movable plates with fully relaxed in-plane restrains at the edges (SS2 or CC2) have lowest flexural vibration frequencies. The nonlinear frequencies of two additional simply supported boundary conditions (SS3 and SS4) with partially relaxed in-plane

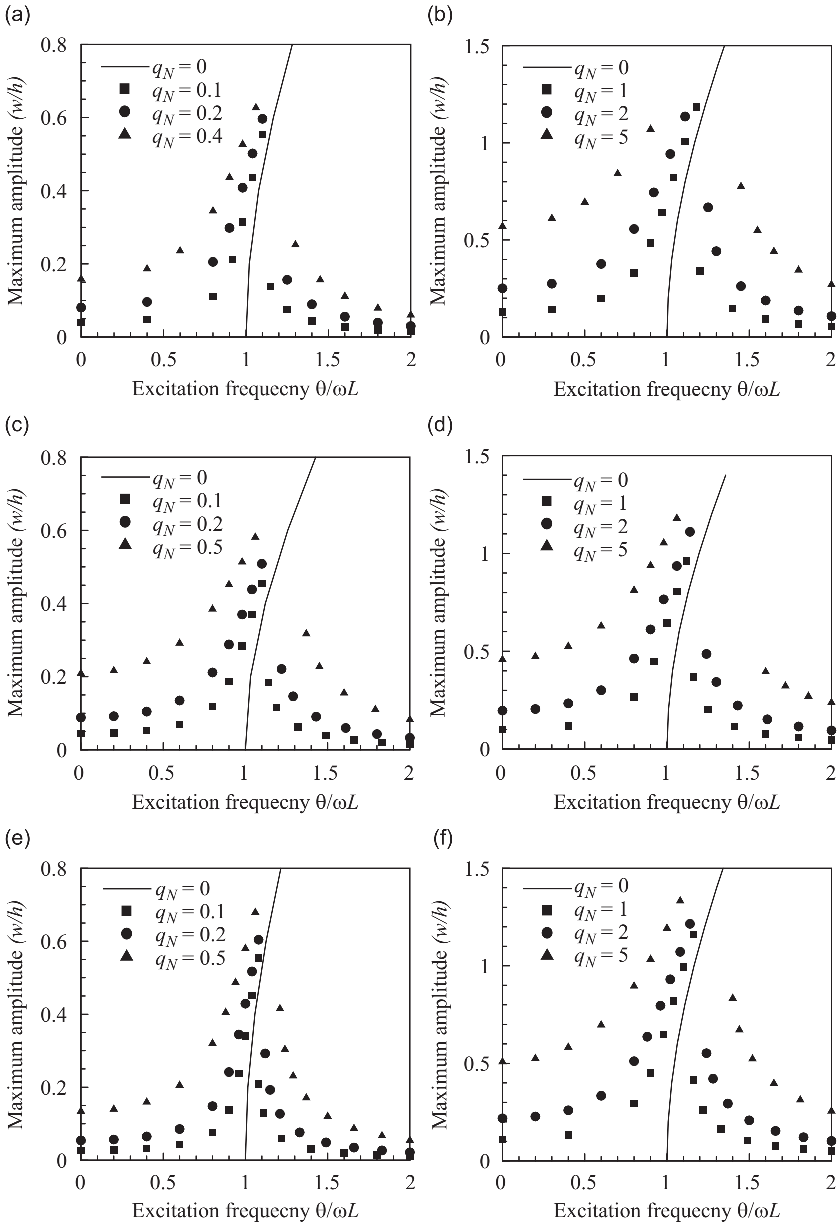

Fig. 2. Nonlinear flexural vibration amplitudes of immovable thin (a/h=100) square plates under uniformly distributed transverse harmonic pressure $q_0 \sin \theta t$: (a) isotropic simply supported (SS1), (b) isotropic clamped (CC1), (c) cross-ply simply supported (SS1), (d) cross-ply clamped (CC1), (e) angle-ply simply supported (SS1), (f) angle-ply clamped (CC1). ($q_N = q_0 a^3/D$ for isotropic plate and $q_N = q_0 a^3/E_T t^3$ for composite plate).

constraints at the edges are also studied in Table 2. This increase of hardening nonlinearity with the increase of in-plane restrains at the edges is also observed for cross-ply [0°/90°/0°/90°/0°] and angle-ply [45°/−45°/45°/−45°/45°] square composite plates. It is observed that in general, the degree of hardening nonlinearity is more for cross-ply plates compared to angle-ply plates for immovable boundary condition.

4.2. Nonlinear forced vibration under transverse harmonic pressure $q(t) = q_0 \sin\theta t$

Now, the forced vibration characteristics of a movable simply supported (SS2) isotropic square plate (a/h=100) under transverse harmonic pressure $q_0 \sin\theta t$ (θ is in the vicinity of linear vibration frequency ω_L; q_0=0.20D/a^3; $D = Eh^3/12(1-\nu^2)$) are taken-up for investigation. The forced vibration amplitudes $\{\delta_m, \delta_b\}^{\mathrm{T}}$ are obtained from the matrix amplitude Eq. (6). Time history analysis with Newmark's time integration technique is carried out starting from the initial condition ($\delta = \{\delta_m, \delta_b\}^{\mathrm{T}}$ at time t=T/4) and the dynamic response of transverse displacement (w_c/h) at the center and in-plane displacement (100u_c) at the mid-point of the edges are presented in Fig. 1 for different excitation frequencies (θ=0.6ω_L, 0.8ω_L, ω_L and 1.2ω_L). The dynamic response is observed to be steady-state and hence the validity of matrix-amplitude Eq. (6) is established for the case of transverse vibration. Further, the steady-state vibration at excitation frequency θ=1.2ω_L is observed to be out-of-phase with the applied load corresponding to the flexural motion $\delta(t) = \{u_0 \sin^2\theta t,\ v_0 \sin^2\theta t,\ w\sin(\pi+\theta)t,\ \gamma_{xz}\sin(\pi+\theta)t,\ \lambda_{yz}\sin(\pi+\theta)t\}^{\mathrm{T}}$.

Next, the nonlinear forced vibration amplitudes (w/h) of immovable simply supported (SS1) and clamped (CC1) plates under transverse harmonic pressure $q_0 \sin\theta t$ are studied Fig. 2. The backbone curves, i.e., the frequency-amplitude relationships, for the case of free flexural vibration (q_0=N_0=0; as reported in Table 2), are represented in Fig. 2 as solid line. The nonlinear forced vibration amplitudes ($w_{\max}/h$) under non-dimensional excitation frequency (θ/ω_L) is presented as scattered symbols for various values of the non-dimensional load parameters $q_N = q_0a^3/D$ and $q_N = q_0a^3/E_Tt^3$ for isotropic and composite plates respectively. It is observed from the figure that, the flexural vibration amplitude (w/h) increases as the excitation frequency (θ) either increases from zero or decreases from a higher value (say, θ=2ω_L). As the excitation frequency approaches the linear flexural vibration frequency (ω_L) of the plate from either side, the nonlinear flexural vibration amplitude increases rapidly (tangential to the backbone curves) as structural damping is not considered in the present study. Further, the vibration at a higher excitation frequency (points on the right side of the backbone curve) is observed to be out-of-phase with the applied load.

4.3. Nonlinear forced vibration under periodic in-plane load $N_d \sin^2\theta t$

Next, the nonlinear dynamic instability characteristics, i.e., the flexural vibration amplitudes of a movable simply supported (SS2) isotropic square plate (a/b=1; a/h=100) under periodic in-plane load of the form $N_d \sin^2\theta t$ (N_d=0.5N_{cr}, where N_{cr} is the static buckling load) are taken up for investigation. The steady-state flexural vibration amplitudes $\{\delta_m, \delta_b\}^{\mathrm{T}}$ are obtained from the matrix amplitude Eq. (6). Similar to the static stability analysis [25], a load perturbation $q_0 \sin\theta t$ (q_0 produces an approximate central displacement of 0.002h) is applied here to initiate bifurcation buckling. Again, time history analysis is carried out starting from the initial condition ($\delta = \{\delta_m, \delta_b\}^{\mathrm{T}}$ at time t=T/4) and the dynamic response of transverse displacement (w_c/h) at the center and in-plane displacement (10u_c) at the mid-point of the edges are presented in Fig. 3 for different excitation frequencies (θ=0.8ω_L, 0.9ω_L, ω_L and 1.1ω_L). The dynamic response is observed to be steady-state and hence the validity of matrix-amplitude Eq. (6) is established again for the case of periodic in-plane load $N_d \sin^2\theta t$.

Now, the steady-state flexural vibration amplitudes (w/h) of movable simply supported (SS2) and clamped (CC2) thin square plates under periodic in-plane load $N_d \sin^2\theta t$ are studied in Fig. 4 for various values of load parameters (N_d=0,

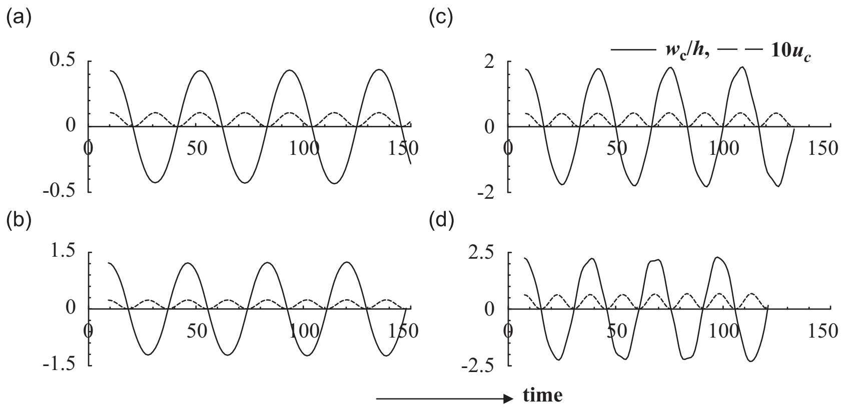

Fig. 3. Steady-state flexural vibration of an isotropic square plate with movable (SS2) simply supported boundary condition under periodic in-plane load $N_d \sin^2\theta t$ (a/h=100; N_d=0.5N_{cr}): (a) θ=0.8ω_L, (b) θ=0.9ω_L, (c) θ=1.0ω_L, (d) θ=1.1ω_L.

$\pm0.25N_{cr}$, $\pm0.5N_{cr}$ and $\pm0.75N_{cr}$). The matrix-amplitude Eq. (6) is solved for both compressive ($+N_d$) and tensile ($-N_d$) load parameters and the corresponding frequency-amplitude curves are obtained on each side of the backbone curve (N_d=0) in Fig. 4. Hence, for a particular load parameter, the left side and right side curves represent steady-state forced vibration amplitude corresponding to excitation load $N_d\sin^2\theta t$ and $-N_d\sin^2\theta t$, respectively. It is observed that with the increase of compressive load parameters (N_d=0.25N_{cr}, 0.5N_{cr} and 0.75N_{cr}), the frequency-amplitude curves shift towards the left side of the backbone curve, indicating a resonance at a lower excitation frequency. The same trend is observed for isotropic, cross-ply [0°/90°/0°/90°/0°] and angle-ply [45°/−45°/45°/−45°/45°] square plates with movable in-plane boundary condition.

Thereafter, the dynamic instability regions of a simply supported (SS2) isotropic square plate under periodic in-plane load $N_s\pm N_d\sin^2\theta t$ (N_d=0.5N_{cr}; θ=0.9ω_L) are studied in Fig. 5. Modified matrix-amplitude Eq. (8) is solved iteratively to obtain the frequency-amplitude relationships for five different values of static in-plane load N_s, i.e., N_s=−0.4N_{cr}, 0.2N_{cr}, 0, 0.2N_{cr} and 0.4N_{cr}. It is observed that, the dynamic instability region shifts towards lower frequency domain (θ/ω_L), when

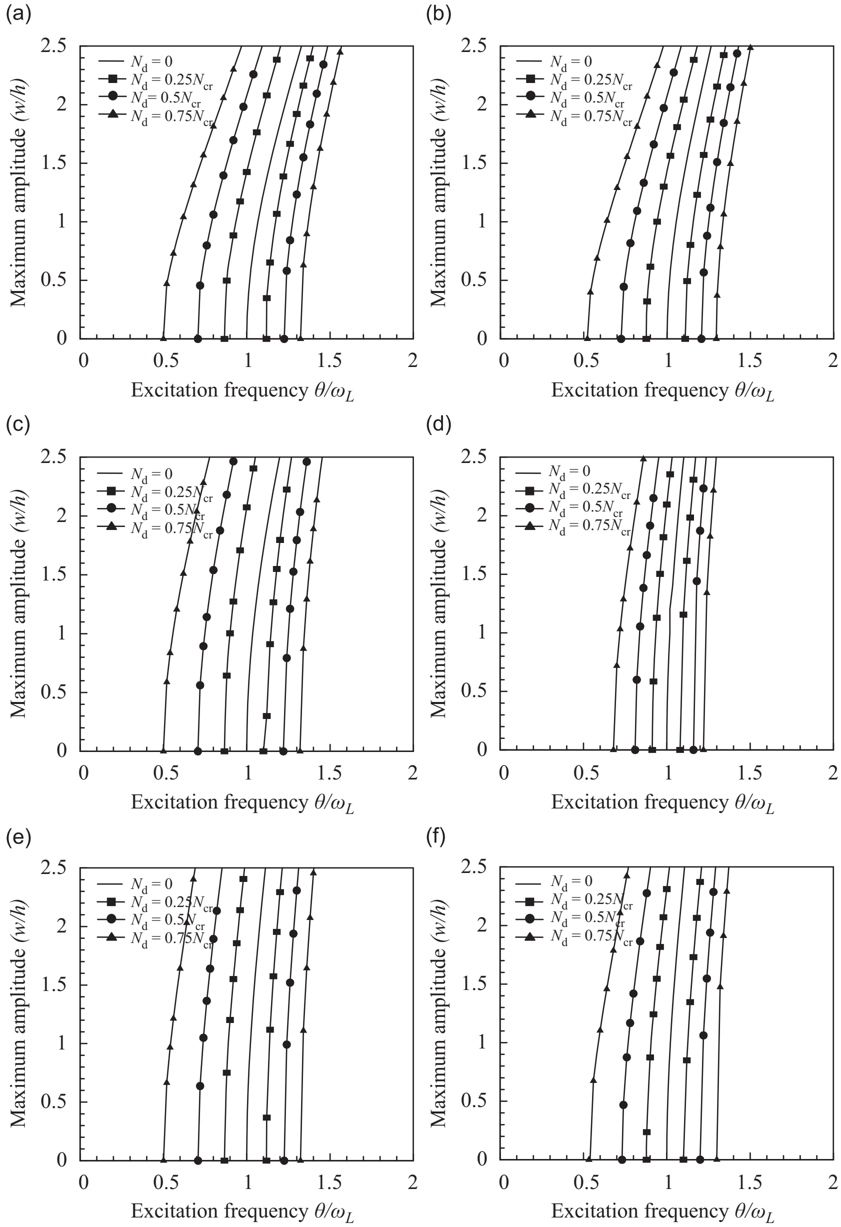

Fig. 4. Nonlinear flexural vibration amplitudes of a movable square plate under periodic in-plane load $\pm N_d\sin^2\theta t$: (a) isotropic simply supported (SS2), (b) isotropic clamped (CC2), (c) cross-ply simply supported (SS2), (d) cross-ply clamped (CC2), (e) angle-ply simply supported (SS2), (f) angle-ply clamped (CC2). ($q_N = q_0a^3/D$ for isotropic plate and $q_N = q_0a^3/E_Tt^3$ for composite plate). Left and right side curves correspond to positive and negative values of load parameter N_d.

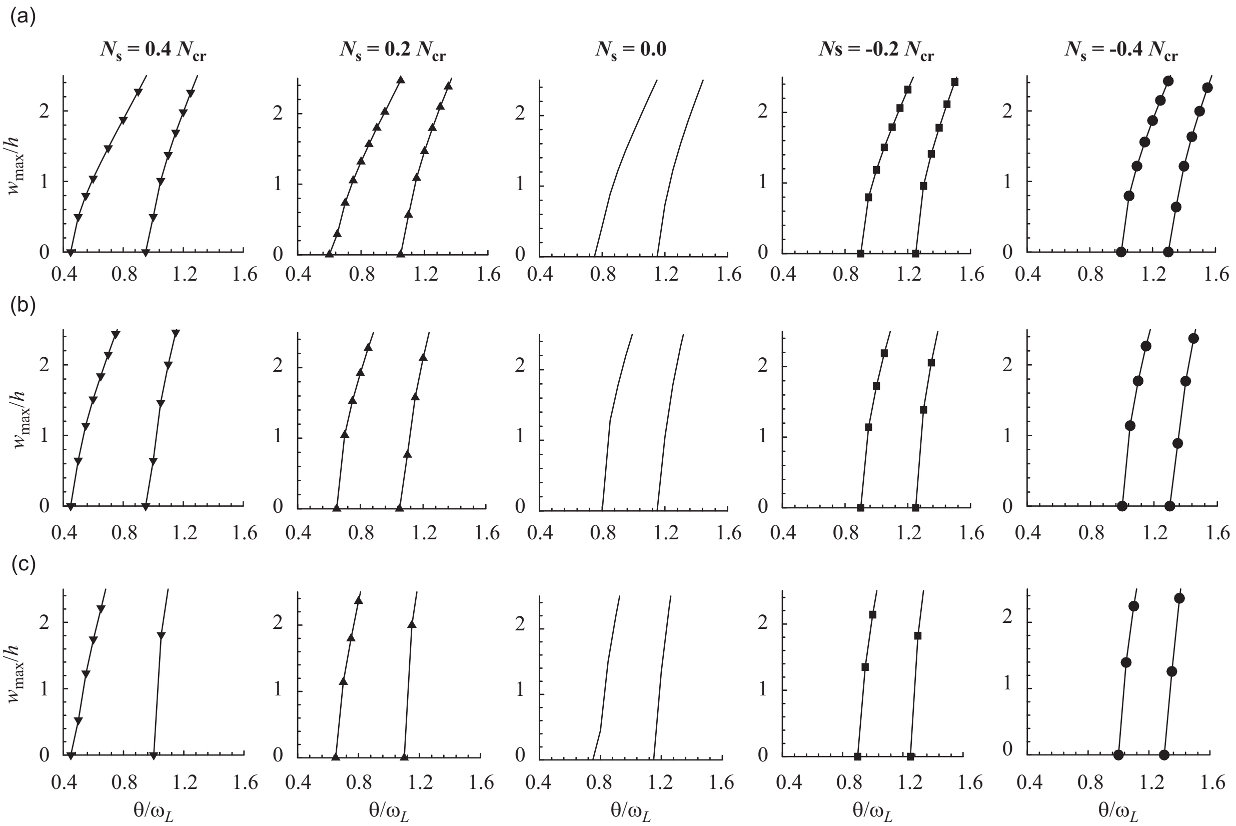

Fig. 5. The Effect of static load (N_s) on nonlinear dynamic stability region of thin square plates (a/h=100; N_d=±0.5N_{cr}): (a) isotropic plate, (b) cross-ply [0°/90°/0°/90°/0] plate, (c) angle-ply [45°/−45°/45°/45°/45°] plate.

the plate is subjected to a static compressive load, and shifts towards right, when the plate is subjected to a static tensile load. Further, time history analysis is carried out with the initial condition $\{\delta_m, \delta_b\}^T$ obtained from the modified matrix-amplitude Eq. (8) and the dynamic response of the central displacement (w_c/h) and in-plane displacement ($50u_c$) at the mid-point of the edges are presented in Fig. 6 along with corresponding phase portrait. It may be observed that the response is approximately steady-state and harmonic. The static tensile load (N_s=−0.4N_{cr}, −0.2N_{cr}) stabilizes the plate, while compressive load (N_s=0.2N_{cr}, 0.4N_{cr}) have a destabilizing effect with higher values of vibration amplitude (w/h).

4.4. Primary dynamic stability under in-plane load $N_1 \cos 2\theta t$

Now, the effect of initial condition on the dynamic response (loss of stability) of a simply supported (SS2) isotropic thin square plate under harmonic in-plane load $N(t) = 0.5N_{cr} \cos 2\theta t$ (N_s=−0.5N_{cr}; N_d=N_{cr}; θ=0.9ω_L) is investigated in Fig. 7. Dynamic response analysis is carried out with different initial conditions (w_0/h) and the variation of central displacement (w_c/h) with time is plotted in the figure along with the corresponding phase portraits. The buildup of vibration, when the initial amplitude is much lower (w_0/h=0.0365 and 0.134) than the steady-state amplitude is presented in Figs. 7(a) and (b). The vibration amplitudes slowly increase with time and reach to their maximum ($w_{\max}/h$=0.924 and 0.888) in about 5 to 6 cycles then slowly reduce in next 5 to 6 cycles to their corresponding minimum values ($w_{\min}/h$=w_0/h=0.0365 and 0.134). This periodical increase and decrease of vibration amplitude showing the existence of beats has been reported earlier by Bolotin [13] and Ganapathi et al. [23,24]. The steady state vibration amplitude as obtained from Eq. (8) is w/h=0.709. Hence, the time history analysis starting from the same initial condition w_0/h=0.709 is approximately steady (Fig. 7c) even though minor oscillation (periodic) of the amplitude is observed ($w_{\max}/h$=0.7102, $w_{\min}/h$=0.6105). This minor oscillation is attributed to the approximation involved in Eq. (8a) and the mismatch in time function of load ($0.5N_{cr} \cos 2\theta t$) and in-plane strains ($\varepsilon = \varepsilon_s + \varepsilon_d \sin^2 \theta t + \varepsilon_{NL} \sin^2 \theta t$). However, when the initial amplitude (w_0/h=1.105 and 1.422) is higher than the steady-state condition, vibration amplitude rapidly decrease to their corresponding minimum ($w_{\min}/h$=0.2602 and 0.5748) and then again increases to the initial amplitude ($w_{\max}/h$=w_0/h) as observed from Fig. 7(d) and (e). The dynamic response of

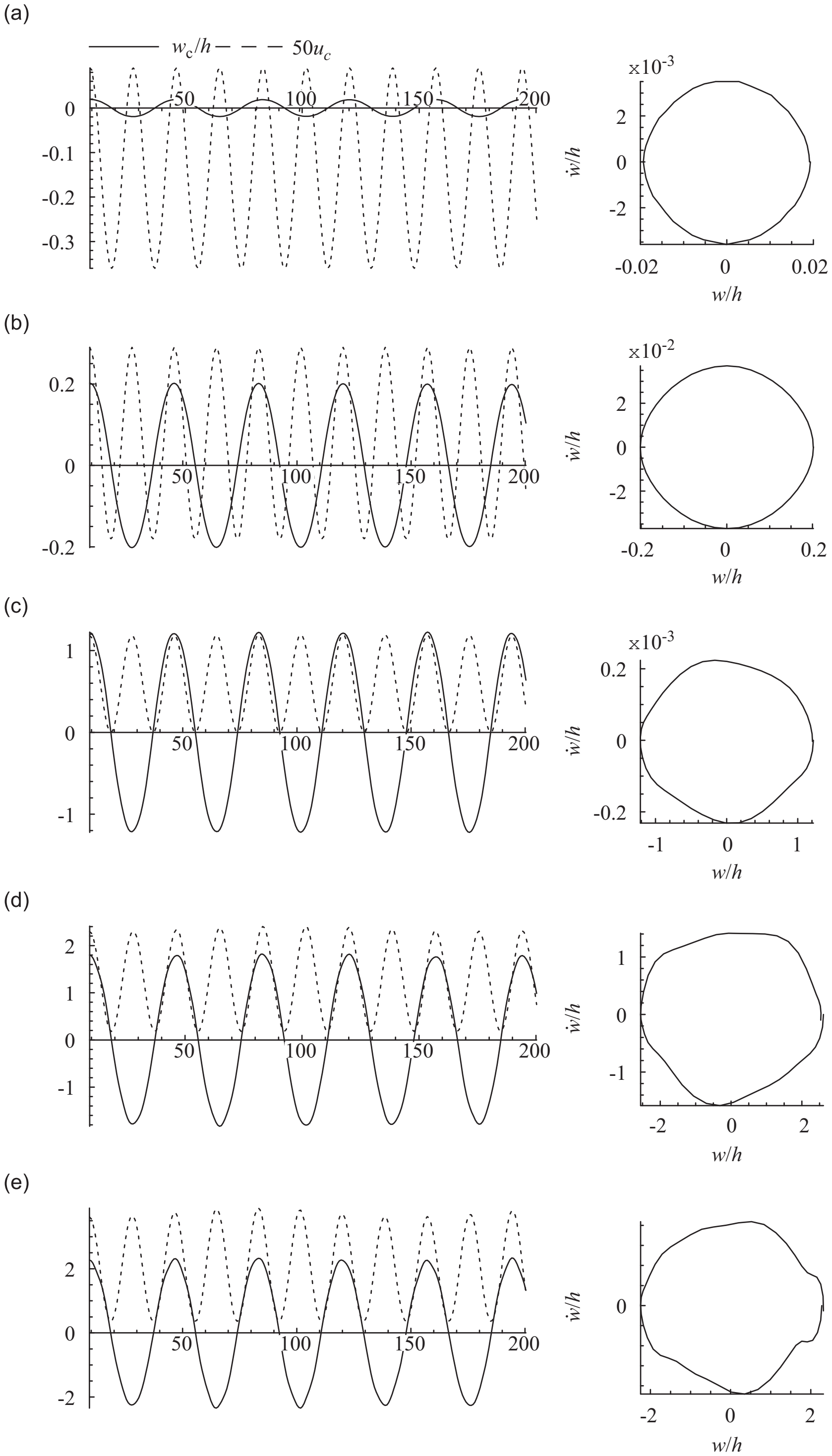

Fig. 6. Steady state dynamic response (time history and corresponding Phase Portrait) of an isotropic square plate under periodic in-plane load $N=N_s+N_d \sin^2 \theta t$ ($N_d=0.5N_{cr}$; $\theta=0.9\omega_L$; $a/h=100$): (a) $N_s=-0.4N_{cr}$, (b) $N_s=-0.2N_{cr}$, (c) $N_s=0$, (d) $N_s=0.2N_{cr}$, (e) $N_s=0.4N_{cr}$.

symmetrically laminated cross-ply [0°/90°/0°/90°/0°] and angle-ply [45°/−45°/45°/−45°/45°] composite square plates excited by harmonic in-plane load $N=0.5N_{cr} \cos 2\theta t$ ($\theta=0.9\omega_L$; $a/h=100$) is studied in Fig. 8 starting from the initial condition as obtained from Eq. (8). Similar to the isotropic plate, the response of cross-ply and angle-ply composite plates are observed to be approximately steady-state.

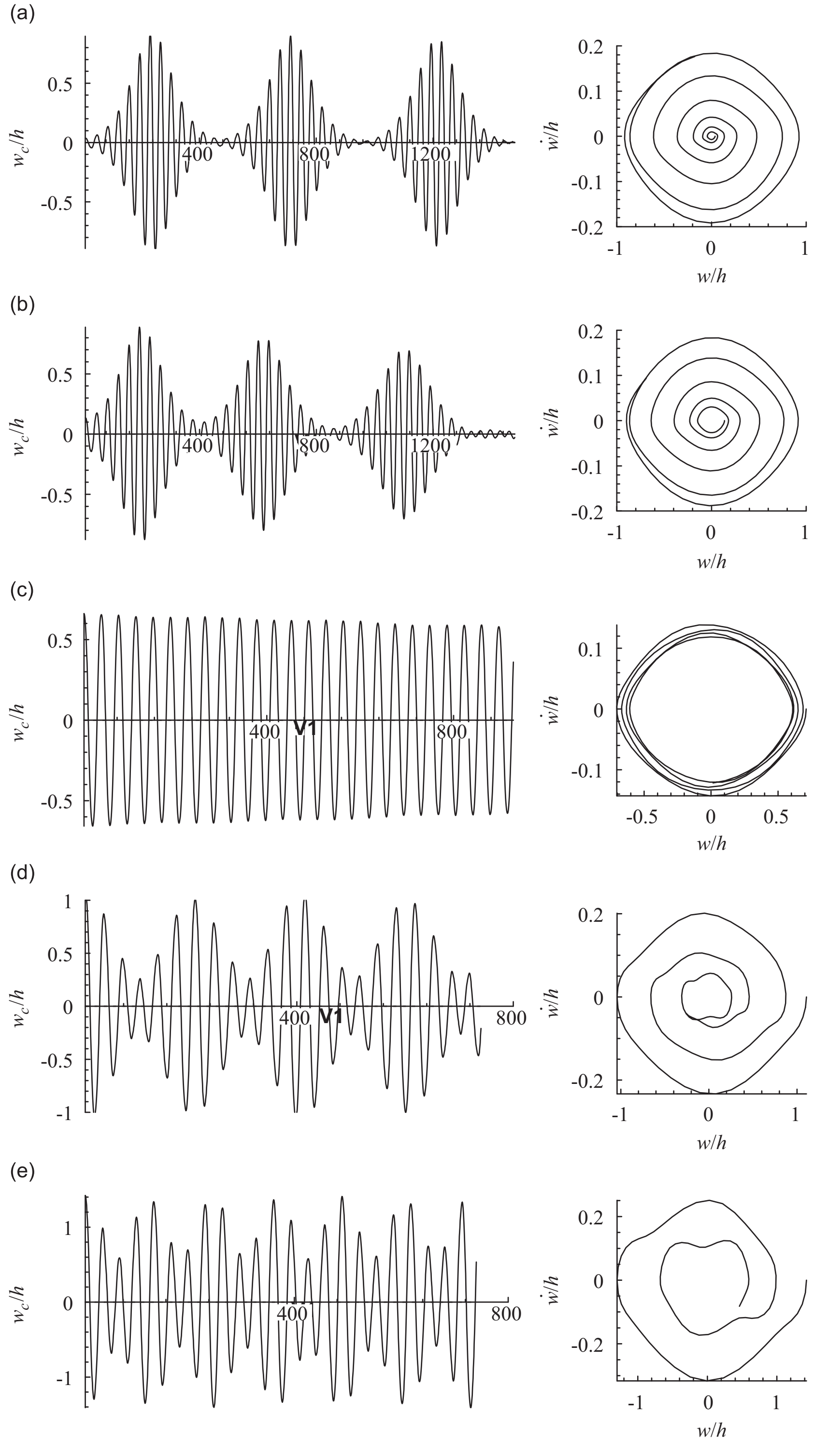

Fig. 7. Effect of initial condition (w_0/h) on the dynamic response (time history and corresponding Phase Portrait) of an isotropic square plate under harmonic in-plane load N=0.5$N_{cr}\cos 2\theta t$ (θ=0.9ω_L; a/h=100): (a) w_0/h=0.0365, $w_{\max}/h$=0.924, (b) w_0/h=0.134, $w_{\max}/h$=0.888, (c) w_0/h=0.709, $w_{\min}/h$=0.6105, (d) w_0/h=1.105, $w_{\min}/h$=0.2602, (e) w_0/h=1.422, $w_{\min}/h$=0.5748.

Now, the buildup of vibration of symmetrically laminated cross-ply [0°/90°/0°/90°/0°] and angle-ply [45°/−45°/45°/−45°/45°] thin (a/h=100) and thick (a/h=20) composite square plates excited by harmonic in-plane load N=0.5$N_{cr}\cos 2\theta t$ (θ=0.9ω_L; a/h=100) is presented in Figs. 9 and 10, respectively. The time history analysis has been started with a small

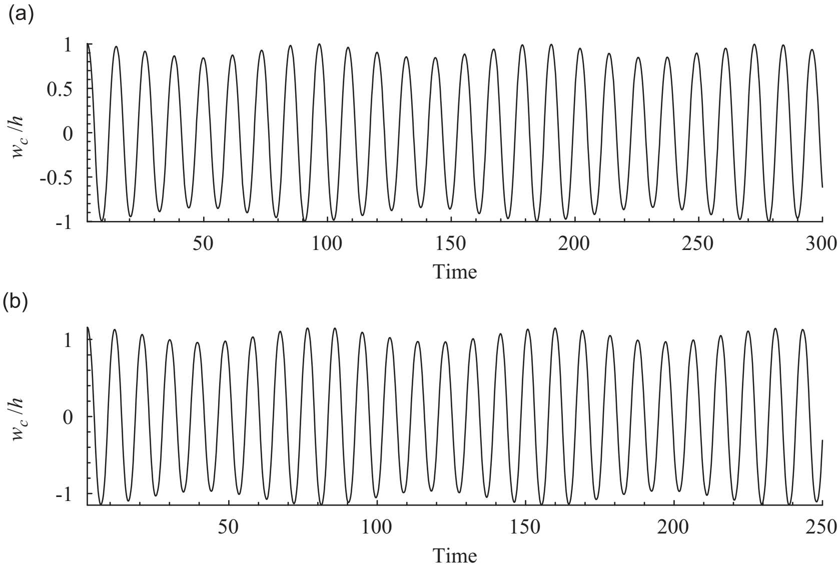

Fig. 8. Dynamic response of symmetrically laminated composite square plates under harmonic in-plane load N=0.5$N_{cr}\cos 2\theta t$ (θ=0.9ω_L; a/h=100), when the initial condition is in the neighborhood of steady-state response: (a) cross-ply [0°/90°/0°/90°/0°]; w_0/h=1.0036, $w_{\max}/h$=1.0036; Beat=8 cycles, (b) angle-ply [45°/−45°/45°/−45°/45°]; w_0/h=1.1586, $w_{\max}/h$=1.1586; Beat=9 cycles.

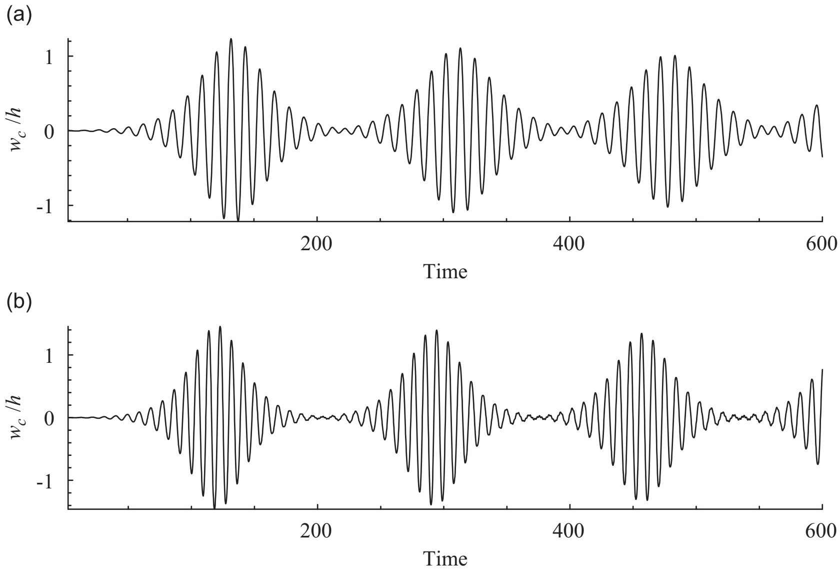

Fig. 9. Buildup of vibration of symmetrically laminated composite thin square plates under harmonic in-plane load N=0.5$N_{cr}\cos 2\theta t$ (θ=0.9ω_L; a/h=100): (a) cross-ply [0°/90°/0°/90°/0°]; w_0/h=0.003; $w_{\max}/h$=1.238; Beat=13 cycles, (b) angle-ply [45°/−45°/45°/−45°/45°]; w_0/h=0.002, $w_{\max}/h$=1.456; Beat=15 cycles.

initial disturbance to initiate bifurcation buckling. Similar to the case of isotropic square plate, the vibration amplitude is observed to increase slowly to the maximum and then decreases slowly to the initial amplitude. This periodic increase and decrease of vibration amplitude takes about 13–15 cycles for composite plates.

It may be pointed out here that, the buildup of vibration from lower amplitude or from rest (w_0/h approximately zero) is of much practical significance to the designer. The maximum vibration amplitude in this case is observed to be 30 percent higher than the corresponding steady-state condition.

5. Conclusions

Large amplitude flexural vibration characteristics of isotropic and composite square plates under transverse harmonic pressure or periodic in-plane load are investigated here using a shear deformable finite element approach. The element

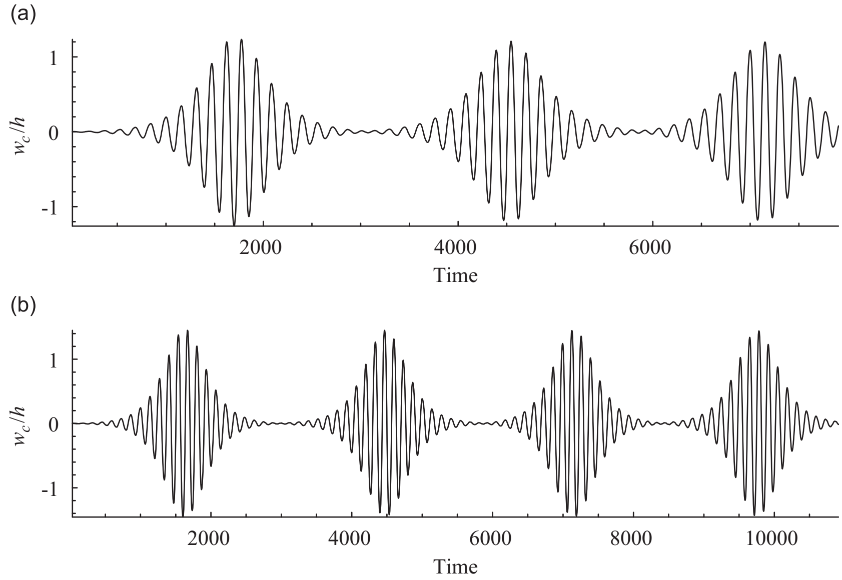

Fig. 10. Buildup of vibration of symmetrically laminated composite thick square plates under harmonic in-plane load N=0.5$N_{cr}\cos 2\theta t$ (θ=0.9ω_L; a/h=20): (a) cross-ply [0°/90°/0°/90°/0°]; w_0/h=0.003, (b) angle-ply [45°/−45°/45°/−45°/45°]; w_0/h=0.002.

employed here has good convergence property. A harmonic solution to the forced vibration problem is assumed and the matrix-amplitude equation is obtained employing Galerkin's method. The proposed matrix amplitude Eq. (6) is quite accurate to predict the steady-state forced vibration amplitudes of symmetrically laminated composite plates under transverse harmonic pressure $q_0 \sin\theta t$ and periodic in-plane load $N_d \sin^2\theta t$. However, minor oscillation in the vibration amplitude is observed for Eq. (8) due to the mismatch in time function of applied load and in-plane strain $(\varepsilon_d + \varepsilon_{NL})\sin^2\theta t$.

In general, nonlinear frequency increases with the increase in vibration amplitude. This degree of hardening type of nonlinearity is more for immovable in-plane boundary conditions compared to those of movable in-plane boundary conditions. The buildup of vibration for a composite plate under periodic in-plane load will be useful to the designer working in the area of nonlinear dynamics of composite plates.

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